TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 196, 1974

ANALYTIC CAPACITY, HÖLDER CONDITIONS, AND τ-SPIKES

BY

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ABSTRACT. We consider the uniform algebra $R(X)$, for compact $X \subset \mathbf{C}$, in relation to the condition $I_{p+\alpha} = \sum_1^\infty 2^{(p+\alpha+1)n}\gamma(A_n(x)\backslash X) < +\infty$, where $0 \le p \in \mathbf{Z}$, $0 < \alpha < 1$, γ is analytic capacity, and $A_n(x)$ is the annulus $\{z \in \mathbf{C}: 2^{-n-1} < |z-x| < 2^{-n}\}$. We introduce the notion of τ-spike for $\tau > 0$, and show that $I_{p+\alpha} = +\infty$ implies x is a $p+\alpha$-spike. If $\overset{\circ}{X}$ satisfies a cone condition at x, and $I_{p+\alpha} < +\infty$, we show that the pth derivatives of the functions in $R(X)$ satisfy a uniform Hölder condition at x for nontangential approach. The structure of the set of non-τ-spikes is examined and the results are applied to rational approximation. A geometric question is settled.

1. For a compact subset X of the Riemann sphere Σ, $R(X)$ denotes the uniform closure on X of the collection $R_0(X)$ of rational functions with poles off X. $R(X)$ is a Banach algebra with respect to the uniform norm $\|\cdot\|_X$ on X. For a positive integer p, $R(X)$ is said to admit a pth order *bounded point derivation* at a point $x \in X$ if the linear functional on $R_0(X)$ defined by $f \mapsto f^{(p)}(x)$ (=the pth derivative of f at x) extends to a continuous linear functional on $R(X)$, i.e., if

$$\sup\{|f^{(p)}(x)| : f \in R_0(X),\ \|f\|_X \le 1\} < +\infty.$$

Hallstrom [4] characterised the points of X at which pth order bounded point derivations exist in terms of analytic capacity, γ. If $U \subset \mathbf{C}$ is a bounded open set we define

$$\gamma(U) = \sup\{|f'(\infty)| : f \in R(\Sigma\backslash U),\ \|f\|_{\Sigma\backslash U} \le 1\}$$

and denote for $x \in \mathbf{C}$, $n \in \mathbf{Z}$, $r \in \mathbf{R}$,

$$A_n(x) = \{z \in \mathbf{C} : 2^{-n-1} < |z-x| < 2^{-n}\},$$
$$U(x,r) = \{z \in \mathbf{C} : |z-x| < r\},$$
$$B(x,r) = \{z \in \mathbf{C} : |z-x| \le r\}.$$

Received by the editors May 23, 1973 and, in revised form, October 9, 1973.
AMS (MOS) subject classifications (1970). Primary 30A82, 46J10, 46J15.
Key words and phrases. Uniform norm, analytic capacity, bounded point derivation, peak point, Gleason metric, potential.
(1) Research supported in part by the National Science Foundation under grant NSF GP-28574(X01).

Hallstrom's theorem. *Let X be a compact subset of* $\mathbf{C}$, $x \in X$, $0 < p \in \mathbf{Z}$. *Then $R(X)$ admits a pth order bounded point derivation at x if and only if*

$$\sum_{n=1}^{+\infty} 2^{(p+1)n}\gamma(A_n(x)\backslash X) < +\infty.$$

This is an extension of sorts of Mel'nikov's theorem [2] characterising the *peak points* for $R(X)$. A point $x \in X$ is said to be a peak point for $R(X)$ if there is a function $f \in R(X)$ such that $f(x) = 1$ and $|f(z)| < 1$ for every $z \in X\backslash\{x\}$.

Mel'nikov's theorem. *Let $X \subset \mathbf{C}$ be compact, $x \in X$. Then x is a peak point for $R(X)$ if and only if*

$$\sum_{n=1}^{+\infty} 2^{n}\gamma(A_n(x)\backslash X) = +\infty.$$

Thus the condition of Mel'nikov's theorem corresponds to that of Hallstrom's, with p replaced by 0. For convenience let us say that $R(X)$ admits a 0th order bounded point derivation at x if x is a nonpeak point.

A. Browder asked what might be the significance for $R(X)$ of Hallstrom's condition for nonintegral p. That is, if $0 < \lambda \in \mathbf{R}$, what does the condition

$$I_\lambda(X, x) = \sum_{n=1}^{+\infty} 2^{(\lambda+1)n}\gamma(A_n(x)\backslash X) = +\infty$$

tell us about the function-theoretic properties of $R(X)$ near x? The idea is that this condition should be related to some kind of λth derivative at x of the functions in $R(X)$.

2. For $0 \leq p \in \mathbf{Z}$, the pth order *Gleason metric* d^p on X is defined by

$$d^p(x, y) = \sup\{|f^{(p)}(x) - f^{(p)}(y)| : f \in R_0(x),\ \|f\|_X \leq 1\},$$

whenever $x, y \in X$. Note that $d^p(x, y)$ may be $+\infty$. This metric was studied in [7], from the point of view of determining for a point $x \in \partial X$ whether there exists a sequence of points $y_n \to x$, $y_n \in \overset{\circ}{X}$, $y_n \neq x$, such that $d^p(y_n, x) \to 0$. In particular, the following things are true [7, Corollary 1, Corollary 3]: *Suppose $\overset{\circ}{X}$ satisfies a cone condition at x, i.e. there is a triangle in $\overset{\circ}{X} \cup \{x\}$ with vertex at x, and Γ denotes the midline of the triangle. Let $0 \leq p \in \mathbf{Z}$. Then, if $R(X)$ admits a pth order bounded point derivation at x, it follows that $d^p(y, x) \to 0$ as $y \to x$, $y \in \Gamma$. If $R(X)$ admits a $(p+1)$st order bounded point derivation at x, then there is a constant $\kappa > 0$ such that $d^p(y, x) \leq \kappa|y - x|$, whenever $y \in \Gamma$.* Abbreviating $I_\lambda = I_\lambda(X, x)$, and combining these facts with the theorems of Mel'nikov and Hallstrom, we deduce that $I_p < +\infty$ implies $d^p(y, x) \to 0$ for $y \in \Gamma$, and $I_{p+1} < +\infty$

implies $d^p(y, x) \le \kappa|y - x|$ for $y \in \Gamma$, so a reasonable guess is that $I_{p+\alpha} < +\infty$ should imply a condition $d^p(y, x) \le \kappa|y - x|^\alpha$.

Theorem 1. *Suppose $X \subset \mathbf{C}$ is compact, $x \in X$, $\overset{\circ}{X}$ satisfies a cone condition at x, Γ is the midline of a sector C with vertex x which lies in $\overset{\circ}{X} \cup \{x\}$, $0 \le P \in \mathbf{Z}$, $0 < \alpha < 1$, and $I_{p+\alpha} < +\infty$. Then there is a constant $\kappa > 0$ such that*

$$d^p(y, x) \le \kappa|y - x|^\alpha \tag{1}$$

whenever $y \in \Gamma$.

Proof. We may suppose $x = 0$, $\Gamma = [-1, 0]$, $C = \{z \in \mathbf{C}: |z| \le 1, |\arg(\pi - z)| \le \alpha\}$ for some $\alpha > 0$. Observe that it suffices to produce a κ such that (1) holds for $y \in [-\frac{1}{2}, 0]$, for given such a κ, (1) then holds with κ replaced by

$$\max\{\kappa, \sup\{d^p(y, x)|y - x|^{-\alpha}: -1 \le y \le -\tfrac{1}{2}\}\}.$$

Fix $y \in [-\frac{1}{2}, 0]$, $f \in R_0(X)$.

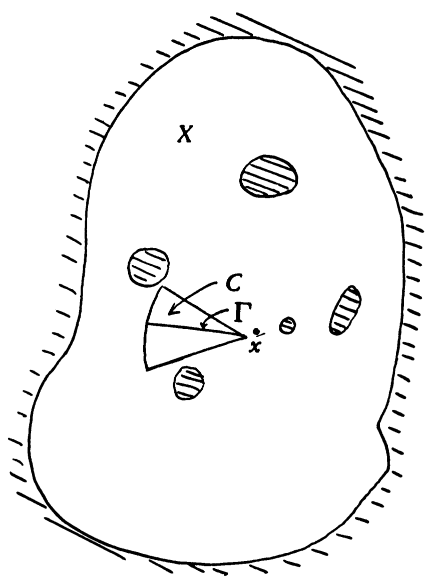

Figure 1

There exists a positive integer N such that f is analytic on $B = B(0, 2^{-N-1})$. Hence

$$f^{(p)}(y) - f^{(p)}(x) = \frac{p!}{2\pi i} \oint_{\partial(B \cup C)} f(z)\{(z-y)^{-(p+1)} - (z-x)^{-(p+1)}\} dz$$

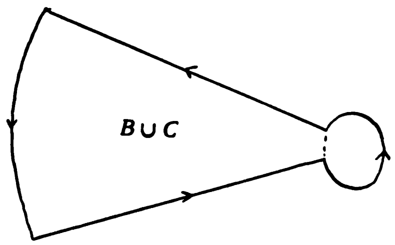

Figure 2

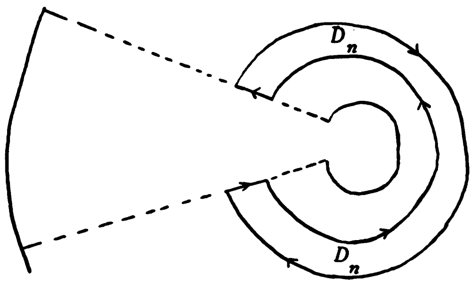

Figure 3

$$= \frac{p!}{2\pi i} \sum_{n=1}^{N} \oint_{\partial D_n} f(z)\{\cdot\} dz + \frac{p!}{2\pi i} \oint_{|z|=1} f(z)\{\cdot\} dz,$$

where $D_n = A_n(0) \setminus C$, and in the integral the orientation of ∂D_n is that which leaves D_n on the right.

Select $q \in \mathbf{Z}$, $q > 1$ such that $y \in A_q(0)$. There is a constant $r > 0$ such that $|\zeta - x| \le r|\zeta - y|$ for all $\zeta \notin C$ (and r may be chosen independent of $y \in [-\frac{1}{2}, 0]$). Hence, if $q + 2 \le n \le N$, $\zeta \in D_n$, then

$$|\{\cdot\}| = \frac{(x-y) \sum_{m=0}^{p} \binom{p}{m} (\zeta - x)^m (\zeta - y)^{p-m}}{(\zeta - x)^{p+1} (\zeta - y)^{p+1}}$$

$$\le |x - y| \sum_{m=0}^{p} \binom{p}{m} |\zeta - x|^{m-p-1} |\zeta - y|^{-m-1}$$

$$\le |x - y|^{\alpha} \sum_{m=0}^{p} \binom{p}{m} |\zeta - x|^{m-p-1} 2^{1-\alpha} |\zeta - y|^{-m-\alpha}$$

$$\text{(since } |y - x| \le 2|\zeta - y|)$$

$$\le |x - y|^{\alpha} \sum_{m=0}^{p} \binom{p}{m} r^{m+\alpha} 2^{1-\alpha} |\zeta - x|^{-p-x-1}$$

$$\le |x - y|^{\alpha} r^{\alpha} (1 + r)^p 2^{1-\alpha} 2^{(p+\alpha+1)n}.$$

If $1 \le n \le q - 2$, $\zeta \in D_n$, then

$$|\{\cdot\}| \le |x - y| \sum_{m=0}^{p} \binom{p}{m} |\zeta - x|^{m-p-1} |\zeta - y|^{-m-1} \le |x - y| \sum_{m=0}^{p} \binom{p}{m} r^{m+1} |\zeta - x|^{-p-2}$$

$$\le |x - y| r (1 + \alpha)^p 2^{(p+2)n} \le |x - y|^{\alpha} r (1 + r)^p 2^{(p+\alpha+1)n}.$$

If $n = q - 1$, q, or $q + 1$, $\zeta \in D_n$, then

$$|\{\cdot\}| \le |x-y| \sum_{m=0}^{p} \binom{p}{m} |\zeta - x|^{m-p-1} |\zeta - y|^{-m-1} \le |x-y| \sum_{m=0}^{p} \binom{p}{m} \tau^{m+1} 2^{(p+2)(q+2)}$$

$$\le |x-y|^{\alpha} \tau (1+\tau)^p 4^{p+2} 2^{(p+\alpha+1)n} \le |x-y|^{\alpha} \tau (1+\tau)^p 8^{p+2} 2^{(p+\alpha+1)n}.$$

Thus, taking λ to be the largest of the numbers $\tau^{\alpha}(1+\tau)^p 2^{1-\alpha}$, $8^{p+2}\tau(1+\tau)^p$, we have $|\{\cdot\}| \le \lambda |x-y|^{\alpha} 2^{(p+\alpha+1)n}$ wherever $1 \le n \le N$, $\zeta \in D_n$. Now, applying the Mel'nikov integral estimate [5], [9], [2] to the (pairwise similar) regions D_n, there is a constant $L > 0$ such that

$$\left| \int_{\partial D_n} g(\zeta)\, d\zeta \right| < L \|g\|_{D_n} \gamma(U \cap D_n)$$

where $g \in R(D_n \setminus U)$, $n = 1, 2, 3, \cdots$. Thus

$$|f^{(p)}(y) - f^{(p)}(x)|$$

$$\le \frac{p!}{2\pi} \sum_{n=1}^{N} \|f\|_X \lambda \cdot L \cdot |x-y|^{\alpha} 2^{(p+\alpha+1)n} \gamma(D_n \setminus X) + \frac{p!}{2\pi} \|f\|_X |x-y| 2^{2p+1}$$

$$\le \frac{p!}{2\pi} \left\{ \lambda \cdot L \sum_{n=1}^{+\infty} 2^{(p+\alpha+1)n} \gamma(A_n(0) \setminus X) + 2^{2p+\alpha} \right\} \|f\|_X |x-y|^{\alpha}.$$

Thus (1) holds with

$$\kappa = \frac{p!}{2\pi} \left\{ \lambda \cdot L \sum_{n=1}^{+\infty} 2^{(p+\alpha+1)n} \gamma(A_n(0) \setminus X) + 2^{2p+\alpha} \right\}.$$

In plain language the conclusion of Theorem 1 is that for nontangential approach to x from $\overset{\circ}{X}$, the pth derivatives of the functions in $R(X)$ satisfy a uniform Hölder condition: $|f^{(p)}(x) - f^{(p)}(y)| \le \kappa \|f\|_X |x-y|^{\alpha}$, where κ is independent of f and y.

3. Wilken [11] observed that $R(X)$ *admits a pth order bounded point derivation at* x $(p \ge 1)$ *if and only if* x *has a representing measure* μ *on* $R(X)$ *such that* $\mu^p(x) < +\infty$. (Recall that a complex Radon measure μ *represents* x on $R(x)$ if $\int f\, d\mu = f(x)$ whenever $f \in R(X)$; and for $0 < \beta \in \mathbf{R}$ the *potential of order* β, μ^{β}, of μ is the function defined by $\mu^{\beta}(z) = \int d|\mu|(\zeta)/|\zeta - z|^{\beta}$ for $z \in \mathbf{C}$; here $|\mu|$ denotes the total variation measure of μ.) This provides us with a second natural way of interpolating between p and $p+1$. In these terms we obtain a result in the opposite direction to Theorem 1, but in a more general setting.

Theorem 2. *Suppose* $X \subset \mathbf{C}$ *is compact,* $x \in X$, $0 \le p \in \mathbf{Z}$, $0 < \alpha < 1$, *and* $I_{p+\alpha} = +\infty$. *Then* $\mu^{p+\alpha}(x) = +\infty$ *whenever* μ *is a representing measure for* x *on* $R(X)$.

Proof. For convenience, suppose $\operatorname{diam} X \leq \frac{1}{2}$. There are two cases to consider.

Case 1°. $\limsup_{n\to\infty} 2^{(p+\alpha+1)n}\gamma(A_n(x)\backslash X) = 0$, so that, for some integer N_0, all the terms beyond the N_0th are bounded by 1. Fix $N_0 < N \in \mathbf{Z}$ and choose $M \geq N$, $M \in \mathbf{Z}$ such that

$$1 \leq \sum_{n=N}^{M} 2^{(p+\alpha+1)n}\gamma(A_n(x)\backslash X) \leq 2.$$

For each $n \in \mathbf{Z}$ with $N \leq n \leq M$ choose $f_n \in R(X \cup (\Sigma\backslash A_n))$ such that $\|f_n\|_\Sigma \leq 1$, $f_n(\infty) = 0$, $f_n'(\infty) > \frac{1}{2}\gamma(A_n(x)\backslash X)$. Form $g_N(z) = |z-x|^\alpha(z-x)^{p+1}\sum_{n=N}^{M} 2^{(p+\alpha+1)n} f_n(z)$. Then a familiar type of argument (cf. [2, p. 206]) shows that the sequence $\{g_N\}_1^\infty$ is uniformly bounded on any bounded set. Defining $h_N(z) = |z-x|^{-\alpha}(z-x)g_N(z)$, we see that $\{h_N\}_1^\infty$ is bounded on bounded sets, and since h_N is analytic on $\Sigma\backslash B(x, 2^{-N})$ we deduce that a subsequence (again denoted $\{h_N\}$) converges pointwise on $\mathbf{C}\backslash\{x\}$ to a function h which is analytic on $\mathbf{C}\backslash\{x\}$. Since h is bounded near x, h is entire. Letting $k_N(z) = (z-x)^{-p-2}h_N(z)$, we see that

$$k_n'(\infty) = \lim_{z\to\infty} (z-x)k_N(z) = \sum_{n=N}^{M} 2^{(p+\alpha+1)n} f'(\infty)$$

lies in $[1, 2^{p+\alpha}]$ for each N, hence by passing to a second subsequence we have $k_N'(\infty) \to \beta$ for some $\beta \in [1, 2^{p+\alpha}]$. Thus $\lim_{z\to\infty}(z-x)^{-p-1}h(z) = \beta$, hence $h(z) = \beta(z-x)^{p+1}$ for $z \in \mathbf{C}$, hence $g_N(z)$ tends pointwise boundedly on bounded subsets of $\mathbf{C}$ to $\beta|z-x|^\alpha(z-x)^p$.

Suppose μ is a representing measure for x on $R(X)$ with $\mu^{p+\alpha}(x) < +\infty$. Then $|z-x|^{-\alpha}(z-x)^{-p}\mu$ is a finite measure and, setting $l_N(z) = |z-x|^{-\alpha}g_N(z)$, l_N is analytic near x, $l_N \in R(X)$,

$$0 = l_N^{(p)}(x) = p!\int \frac{l_N(z)}{(z-x)^p}\,d\mu(z) = p!\int \frac{g_N(z)}{|z-x|^\alpha(z-x)^p}\,d\mu(z)$$

$$\to p!\int \beta\,d\mu(z) = \beta\cdot p!.$$

This is a contradiction.

Case 2°. $\limsup_{n\to+\infty} 2^{(p+\alpha+1)n}\gamma(A_n(x)\backslash X) > 2S > 0$. Let $\{N_i\}_1^\infty$ be a sequence of integers such that

$$2^{(p+\alpha+1)N_i}\gamma(A_{N_i}(x)\backslash X) > 2S,$$

and for each i choose $f_i \in R(X \cup (\Sigma\backslash A_{N_i}))$ such that $\|f_i\|_\Sigma \leq 1$, $f_i(\infty) = 0$,

$f_i'(\infty) = S2^{-(p+\alpha+1)N_i}$. Then, defining $g_i(z) = |z-x|^\alpha (z-x)^{p+1} 2^{(p+\alpha+1)N_i} f_i(z)$, the argument of Case 1° goes through with these new g_i's, and again we arrive at a contradiction.

4. Let us say that x is a *τ-spike* for $R(X)$ if $\mu^\tau(x) = +\infty$ whenever μ represents x on $R(X)$. A peak point is a τ-spike for every $\tau > 0$.

Corollary 1. *Suppose $\overset{\circ}{X}$ satisfies a cone condition at x, Γ is a straight line in $\overset{\circ}{X} \cup \{x\}$ which is not tangential to ∂X at x, $0 \le p \in \mathbf{Z}$, $0 < \alpha < 1$, and x is not a $(p+\alpha)$-spike for $R(X)$. Then there is a constant $\kappa > 0$ such that $d^p(x, y) \le \kappa|x-y|^\alpha$ for $y \in \Gamma$.*

Proof. Combine Theorem 1 and Theorem 2.

5. Next, we examine the structure of the set of τ-spikes. The case $\alpha = 0$ of the following lemma is due to Browder [1, p. 177].

Lemma. *Suppose μ is a Radon measure with no mass at x, $0 < b \in \mathbf{R}$, and $E^\alpha = \{y \in \mathbf{C}: |x-y|^{1+\alpha}\mu^{1+\alpha}(y) < b\}$. Then E^α has full area density at x, for $0 \le \alpha < 1$.*

Proof. For $r > 0$ let $\nu_r = \mathfrak{L}^2|(B(x, r)\setminus E^\alpha)$ (= area measure restricted to the complement of E^α). Then by the definition of E^α and Fubini's theorem,

$$\mathfrak{L}^2(B(x, r)E^\alpha)b = \|\nu_r\| b \le \int |x-y|^{1+\alpha}\mu^{1+\alpha}(y)\, d\nu_r(y) = \pi r^2 \int G_r(z)\, d|\mu|(z),$$

where

$$G_r(z) = \frac{1}{\pi r^2}\int \frac{|x-y|^{1+\alpha}}{|z-y|^{1+\alpha}}\, d\nu_r(y).$$

It is easy to see that $G_r(z)$ tends pointwise boundedly to zero on $\mathbf{C}\setminus\{x\}$, hence $\lim_{r\to 0}[\mathfrak{L}^2(B(x, r)\setminus E^\alpha)/\pi r^2] = 0$.

We note in passing that by applying the technique of [8, Lemma 2] a much stronger result may be obtained. Let C^β denote the *capacity of order* β: if $E \subset \mathbf{C}$, $0 < \beta \in \mathbf{R}$, then $C^\beta(E) = \sup\{|\nu(\mathbf{C})|: \nu$ is a Radon measure with support in E, $\nu^\beta \le 1\}$. Then, *if μ, x, b, E are as in the lemma, it follows that*

$$\sum_{n=1}^{+\infty} 2^{(1+\alpha)n} C^{1+\alpha}(A_n(x)\setminus E^\alpha) < +\infty.$$

In particular, for $\beta > 1+\alpha$, the β-dimensional density at x of β-dimensional *Hausdorff content* M^β (cf. [8]), restricted to the complement of E^α, is zero.

Corollary 2. *Suppose x is not a peak point for $R(X)$, and $0 < \alpha < 1$. Then* *the set* $\{y \in X: y$ *is not an* α*-spike*$\}$ *has full area density at* x.

Proof. There is a representing measure μ for x with no mass at x [2, p. 54, 11.3]. Applying the lemma with $b = 1$ and $\alpha = 0, \alpha$ respectively we deduce that E^0, E^α, and hence $E^0 \cap E^\alpha$, have full area density at x. Set $\nu = (z-x)\mu$. Then, for $y \in E^0$, $\hat{\nu}(y) \neq 0$ and $\sigma = \hat{\nu}(y)^{-1}(z-y)^{-1}\nu = \hat{\nu}(y)^{-1}(z-x)(z-y)^{-1}\mu$ represents y on $R(X)$ [1, p. 176]. For $y \in E^0 \cap E^\alpha$, $\mu^{1+\alpha}(y) < +\infty$, hence

$$\sigma^{1+\alpha}(y) = |\hat{\nu}(y)|^{-1} \int \frac{|z-x|}{|z-y|^{1+\alpha}}\, d|\mu|(z) \leq |\hat{\nu}(y)|^{-1} \cdot \text{diam } X \cdot \mu^{1+\alpha}(y) < +\infty,$$

so $E^0 \cap E^\alpha$ consists entirely of non-α-spikes.

6. This enables us to strengthen Bishop's criterion [2, p. 54] for $R(X) = C(X)$ (= the space of all continuous functions on X). Bishop showed that *if* $\mathfrak{L}^2$ *almost all points of* X *are peak points for* $R(X)$, *then* $R(X) = C(X)$.

Theorem 3. *Let* $X \subset \mathbf{C}$ *be compact. Then* $R(X) = C(X)$ *if for* $\mathfrak{L}^2$ *almost every* $x \in X$ *there is* α, $0 < \alpha < 1$, *and* x *is an* α*-spike*.

Proof. By Corollary 2, every point of X is a peak point for $R(X)$, hence by Bishop's theorem, $R(X) = C(X)$.

A direct proof is also available: if ν is an annihilating measure for $R(X)$, then $\nu^{1+\alpha}(y) < +\infty$ for $\mathfrak{L}^2$ almost all y; if $\nu^{1+\alpha}(y) < +\infty$ and $\hat{\nu}(y) \neq 0$, then, constructing σ as in the proof of Corollary 2, we see that y is not an α-spike for $R(X)$, hence $\hat{\nu}(y) = 0$ for $\mathfrak{L}^2$ almost all y, hence $\nu = 0$ [2, p. 46, 8.2].

Corollary 3. $R(X) = C(X)$ *if for* $\mathfrak{L}^2$ *almost every* $x \in X$ *there is* α, $0 < \alpha < 1$, *with* $I_\alpha(X, x) = +\infty$.

Proof. Theorem 2 + Theorem 3.

Corollary 4. *Suppose for* $\mathfrak{L}^2$ *almost every* $x \in X$ *there exists* α, $0 < \alpha < 1$, *and*

$$\limsup_{r \to 0} \frac{\gamma(U(x, r) \setminus X)}{r^{1+\alpha}} > 0.$$

Then $R(X) = C(X)$.

This last fact was previously known; in fact it is known that α may be replaced by 1 [2, p. 207]. However, that result depends on the instability of analytic capacity, a very deep theorem. It is not possible to replace α by 1 in Corollary 3, for Wermer [10] has shown that there exist compact sets X such that $R(X)$ admits no bounded point derivations (hence, by Hallstrom, $I_1(X, x) = +\infty$ for all $x \in X$), yet $R(X) \neq C(X)$.

To prove Corollary 4 note that the argument of Case 2° of the proof of Theorem 2 shows that the lim sup condition implies x is an α-spike.

Fix $X \subset \mathbf{C}$, compact, and set $D^\tau = \{x \in X: I_\tau(X, x) < +\infty\}$. In [6] it was noted that D^0 never contains isolated points, while for $\tau \geq 1$, D^τ may consist of a single point. We are now in a position to complete the picture.

Corollary 5. *If* $0 < \tau < 1$, *then* D^τ *has full area density at each of its points.*

Proof. Each point x of D^τ belongs to D^0, hence is a nonpeak point, and by Corollary 2 the set of non-τ-spikes has full area density at x. By Theorem 2 every non-τ-spike is in D^τ.

When Gleason first introduced parts [3] he expressed the hope that there might be bounded (first order) point derivations at most points of a nontrivial part. While this hope was not borne out by the facts, the foregoing discussion shows that at most points of a part of $R(X)$ the functions in $R(X)$ just barely miss being differentiable, in the sense that $\mathfrak{L}^2$ almost all points of a part are not α-spikes for any α in $(0, 1)$.

We should mention that there are examples of points which are α-spikes but not peak points, so that the theory is not vacuous. For instance, consider a *Zalcman set*, a compact set X obtained by deleting from the closed unit disc a sequence of open balls B_n of radius r_n, with $B_n \subset A_n(0)$, $n = 1, 2, 3, \cdots$. Since $\gamma(A_n(0)\setminus X) = \gamma(B_n) = r_n$, Mel'nikov's theorem implies that 0 is a peak point for $R(X)$ if and only if $\sum_{n=1}^{+\infty} 2^n r_n = +\infty$. By Theorem 2, if $0 < \alpha < 1$, then 0 is an α-spike for $R(X)$ provided $\sum_{n=1}^{+\infty} 2^{(1+\alpha)n} r_n = +\infty$. Choose $\beta \in (1, 1+\alpha)$, $r_n = 2^{-(1+\beta)n}$. Then 0 is an α-spike but not a peak point. Incidentally, for a Zalcman set the converse to Theorem 2 is true: 0 is a τ-spike if *and only if* $I_\tau = +\infty$.

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